A note on contestation-based tournament solutions

Scott Moser

Social Choice and Welfare

ISSN 0176-1714
Volume 41
Number 1

Soc Choice Welf (2013) 41:133-143
DOI 10.1007/s00355-012-0672-4

Volume 41 · Number 1 · June 2013

Social Choice and Welfare

ORIGINAL PAPERS

Springer

Springer

Soc Choice Welf (2013) 41:133–143
DOI 10.1007/s00355-012-0672-4

ORIGINAL PAPER

A note on contestation-based tournament solutions

Scott Moser

Received: 11 January 2011 / Accepted: 28 April 2012 / Published online: 25 May 2012

Abstract In tournaments, one alternative *contests* another if is a "winner" among only alternatives that beat it. This paper examines the consequences and limitations of the *contestation relation* by considering a procedure in which alternatives that are contested are iteratively eliminated from consideration. In doing so, a new family of tournament solutions are introduced and related to existing refinements of the Banks set. Findings show that iterated removal of contested alternatives a limited device for choosing from tournaments. These results contrast with results regarding the top-set of the contestation relation. Results highlight the role of the top-set operator for choice from tournaments.

1 Introduction

When considering collective choice from a finite number of alternatives, *tournaments* (directed, complete, asymmetric graphs) have been a central object of study [see Laslier (1997) for a comprehensive treatment of tournaments and majority voting]. In the absence of a Condorcet winner, however, choice from tournaments becomes an involved matter. Central to the problem of group choice from a tournament is cyclic majority preference. One useful way of approaching this problem is via the contestation relation. The contestation relation is a useful way of relating alternatives'

This work benefited greatly from the comments of two anonymous reviewers as well as associate editor Prof. Laslier. Any remaining errors are, of course, mine alone. A significant portion of this work was conducted while I was a Post-doctoral Research Fellow at Nuffield College, University of Oxford, to which I am most grateful.

S. Moser (✉)
Department of Government, University of Texas at Austin, Austin, USA
e-mail: smoser@austin.utexas.edu

ability to "displace from consideration" other alternatives. When group choice from a tournament is viewed as a cooperative process of considering and reconsidering alternatives based on a given criteria, contestation proves a powerful way of formalizing and examining the types of choices groups may make. In the original formulation and investigation of the contestation relation in social choice settings (Schwartz 1990), two related but conceptually distinct notions were employed: the *contestation relation* which is a particular binary relation derived from a tournament solution and the *top-set of the contestation relation*. The top-set is a generalization of the top-cycle and can be applied to general binary relations. Rather than employing the top-set operator to obtain a tournament solution from the contestation relation, this paper instead introduces a family of tournament solutions by iteratively excluding from consideration contested alternatives. The introduced tournament solutions successively refine the Banks set (Banks 1985), but even in the limit remain a superset of the tournament equilibrium set (Schwartz 1990). The introduced tournament solutions are not meant to supplant existing methods; rather, they suggest the "strength" of a variety of tournament solutions constructed from the contestation relation—specifically TEQ (Schwartz 1990) and $\mathring{S}$ and $S^{(n)}$ (Brandt et al. 2010a)—lies not in the contestation relation per se, but rather in the top-set operator.

An extensive general treatment of tournaments can be found in Laslier (1997), who discusses tournaments as a tool for studying social choice under majority voting. Schwartz (1990) introduces the contestation relation and defines an important tournament solution, the tournament equilibrium set (TEQ) in terms of the top-set of the contestation relation, the properties of which have been studied in Dutta (1990), Laffond et al. (1993) and Houy (2009). Some of the most basic properties of TEQ were unknown until the recent discovery of Brandt et al. (2011) who show that TEQ is not pair-wise intersecting and hence is not monotonic and does not satisfy the strong-superset property (Laffond et al. 1993).

Schwartz (1990) motivates the contestation relation, and the derived tournament solution, TEQ, by considering group choice as a process of renegotiation and recontracting. This conceptualization of re-consideration is explored in Schwartz (1990), and may be described as follows. If a group is to cooperatively choose from a tournament, then a potential collective choice—a "tentative contract" in the language of Schwartz (1990)—is subject to re-consideration. If x is proposed as a group's choice, some alternatives might "unseat" x from consideration; specifically, those chosen from alternatives that are majority-preferred to x. If x is to be a collective choice, so should anything that could unseat it from consideration. That is, if collective choice comes about through a process of consideration and reconsideration of alternatives, one may which to require that collective choice be closed under re-consideration. This is operationalized via the top-set operation Schwartz (1990) and gives rise to several tournament solutions.

For example, Brandt et al. (2010a) examines the limits of iterating the top-set operator applied to the contestation relation for general tournament solutions and finds that *any* tournament solution converges (in a sense made precise in their paper) to TEQ. He introduces a family of tournament solutions based on the top-set of contestation and shows convergence of this family and inheritance of some properties.

The present work departs from the above in a number of ways. Most importantly, I consider the iterative deletion of contested alternatives rather than the top-set of the contestation relation. I show that the resulting tournament solutions are found wanting with respect to several desirable properties. The utility of the exercise is hence not to suggest "superior" tournament solutions, but rather to shed light on the use and usefulness of contestation in the absence of the top-set operator. The results, together with Brandt et al. (2010a, 2011) suggest an examination of other binary relations derived from tournament solutions, and the top-sets thereof.

After introducing the basics of tournaments and examples of well-known social choice solutions (Sect. 2), I introduce a family of new tournament solutions in Sect. 3 and show some of their basic properties. Section 4 discusses this research in the context of recent findings, and concludes.

2 Preliminaries

2.1 Tournaments

Many of the definitions and basic results discussed in this section can be found in Laslier (1997).

Definition 2.1 Let X be a non-empty finite set and let and $T \subseteq X^2 = (X \times X)$. The binary relation T is a **tournament** if and only if:

$$\forall x \in X, (x, x) \notin T$$
$$\forall (x, y) \in X^2, x \neq y \Rightarrow [(x, y) \in T \vee (y, x) \in T]$$
$$\forall (x, y) \in X^2, (x, y) \in T \Rightarrow (y, x) \notin T.$$

That is, a tournament is an irreflexive, complete and asymmetric binary relation on a finite set. A tournament may alternatively be defined as a complete asymmetric directed graph, with X being the set of vertices. Denote the set of all tournaments on X by $\mathcal{T}(X)$. I write xTy if $(x, y) \in T$. For any $x \in X$, define $T^{-1}(x) = \{y|yTx\}$ as well as $T(x) = \{y|xTy\}$. For any set X and any subset $Y \subset X$, let $T|_Y$ denote the tournament induced by T on Y.[1] Trivially, if $T \in \mathcal{T}(X)$ and $Y \neq \emptyset$, then $T|_Y$ is a tournament on Y. A set $Y \subseteq X$ is *externally stable* (with respect to T) if $\forall y \in Y, \nexists x \in X \setminus Y$ with xTy.[2]

A major concern of social choice [as well as computer scientists—see Brandt et al. (2009, 2010b)] is to identify "winners" from a given tournament. To that end, define a *tournament solution* as follows:

Definition 2.2 A **tournament solution** on X is a function

$$S : \bigcup_{Y \subseteq X} \mathcal{T}(Y) \rightarrow 2^X$$

[1] More formally, $T|_Y = T \cap (Y \times Y)$, which can be considered as the subgraph of T on Y.

[2] Because T is complete and asymmetric, this is equivalent to : Y is externally stable if for every $z \in X \setminus Y$, there exists $x_z \in Y$ such that $x_z T z$.

for $Y \neq \emptyset$ such that (1) $S(T|_Y) \subseteq Y$ (inclusiveness), (2) $S(T) \neq \emptyset$ for all $T \in \mathcal{T}(X)$ (non-emptiness), (3) relabeling the elements of X does not affect the solution (respect for isomorphisms) and (4) if there is a Condorcet winner, $Cond(T)$, the solution picks it and only it (that is, if there exists an $x \in X$ such that xTy, $\forall y \neq x$, then $S(T) = x = Cond(T)$). If $x \in S(T)$, I write "x is an S-winner of T."

Definition 2.3 A tournament solution $S(\cdot)$ satisfies the **Aïzerman property** if for any tournament T and any $Y \subseteq X$ we have

$$S(T) \subseteq Y \subseteq X \Rightarrow S(T|_Y) \subseteq S(T).$$

Definition 2.4 A tournament solution $S(\cdot)$ satisfies the **strong superset property** (SSP) if for any tournament T and any $Y \subseteq X$ we have

$$S(T) \subseteq Y \subseteq X \Rightarrow S(T|_Y) = S(T).$$

Definition 2.5 A tournament solution S is **monotonic** if for any tournament T, for any $x \in S(T)$ and any tournament T' such that

1. $T'|_{X\backslash\{x\}} = T|_{X\backslash\{x\}}$ and
2. $\forall y \in X, xTy \rightarrow xT'y$

one has $x \in S(T')$.

In words, a solution is monotonic if by adding "support" to a winner it remains a winner.

Given a tournament solution, $S(\cdot)$, one may wish to consider S-winners not only of the entire tournament, T, but of particular subtournaments $T|_Y$ ($Y \subset X$) as well. Specifically, define a binary relation based on S as follows.

Definition 2.6 Let $S(\cdot)$ be a tournament solution and T be a tournament. Define the **contestation relation with respect to** $S(\cdot)$, as the binary relation $D(S, T)$ on X by

$$\forall (x, y) \in X^2,\ xD(S, T)y \text{ iff } x \in S(T|_{T^{-1}(y)}).$$

If $xD(S, T)y$ then we say "x S-contests y," or "x bears on y"—this means that x is an "S-winner" when considering just alternatives that beat y. Clearly, xTy is necessary for $xD(S, T)y$ and hence $D(S, T) \subseteq T$. For a given tournament solution $S(\cdot)$, denote the set of points that S-contest *some* point in X by $D(S, T)^{-1}(X)$.

2.2 The Banks set

Define a tournament solution—the *Banks set*, $B(\cdot)$ —as follows:

Definition 2.7 Let $x \in X$ and $T \in \mathcal{T}(X)$. Then $x \in B(T)$ iff there exists $Y \subseteq X$ such that (1) $T|_Y$ is transitive with x top-ranked and (2) Y is eternally stable (w.r.t. T).

That is, x is in the Banks set if and only if it is top of a transitive, externally stable set. Clearly, if $Cond(T) \neq \emptyset$ then $B(T) = Cond(T)$.

The Banks set (Banks 1985) was originally studied for its non-cooperative interpretation: one may wonder the extent to which voting outcomes can be manipulated when voters are strategic, fully informed of other's preferences, and an agenda setter may order alternatives to be voted on in an amendment agenda. The Banks set is precisely the set of outcomes obtainable in equilibrium to this non-cooperative game. The original characterization of the Banks set was hence in terms of external stability and maximal chains, two concepts useful for the study of "amendability" of voting agendas. However, the Banks set may alternatively be viewed as the collection of alternatives that Banks-contests some alternative. As the following proposition shows, it is the unique tournament solution to satisfy this property.

Proposition 2.8 *(Laslier 1997, Props. 7.1.1, 7.1.2) Let $X \geq 2$. Then $B(\cdot)$ is the unique tournament solution such that*

$$B(T) = D(B, T)^{-1}(X)$$

for all $T \in \mathcal{T}(X)$.

2.3 The tournament equilibrium set

Before introducing the tournament equilibrium set [TEQ, (Schwartz 1990)], it will be useful to review binary relations in general as well as sets that are "externally immune" with respect to a given binary relation.

Definition 2.9 Let R be a binary relation on a set X and let $Y \subseteq X$. Then Y is **retentive for** R (or is "R-retentive") if and only if

$$Y \neq \emptyset \tag{1}$$

$$\nexists (x \in X \setminus Y \text{ and } y \in Y) \text{ such that } xRy. \tag{2}$$

A retentive subset Y is **minimal retentive** for R if $\nexists Z \subset Y$ such that Z is R-retentive.

Retentive sets consist of alternatives that are not "R-ed" by anything not in the set. For a general binary relation R on X, let $Ret_R(X) = \{Y \subseteq X : Y \text{ is } R\text{-retentive}\}$ be the set of all R-retentive sets of X. Following Brandt et al. (2010a), call $Ret_R(X)$ *pairwise intersecting* if and only if $Y, Z \in Ret_R(X) \Rightarrow Y \cap Z \neq \emptyset$. It follows that if Ret_R is pair-wise intersecting, then there exists a unique minimal R-retentive set.[3]

Definition 2.10 The **top-set** of a binary relation R is the union of all minimal retentive subsets of R and is denoted $TS(R)$.

[3] It is well-known that if Y_1 and Y_2 are two retentive subsets for R such that $Y_1 \cap Y_2 \neq \emptyset$ then $Y_1 \cap Y_2$ is retentive for R as well.

The top-set generalizes the the concept of maximal elements—when maximal elements exist, the two concepts are the the same.

Of particular interest will be sets that are $D(S, T)$-retentive, for a given S and T. If a set $Y \subseteq X$ is $D(S, T)$-retentive, then $S(T|_{T^{-1}(y)}) \subseteq Y$ for all $y \in Y$. In an abuse of notation, I take $S(\emptyset) = \emptyset$.[4]

Definition 2.11 (Schwartz 1990) The tournament equilibrium set, TEQ, is a tournament solution such that

$$TEQ(T) = TS(D(TEQ, T))$$

for all $T \in \mathcal{T}$.

That TEQ is non-empty and equals the Condorcet winner when there is one is established in Schwartz (1990, Theorems 1, 2). It is well-known that $TEQ(T) \subseteq B(T)$ for all $T \in \mathcal{T}$ and that $Cond(T) = TEQ(T)$ if $Cond(T) \neq \emptyset$.

TEQ may be motivated by an axiomatic characterization, capturing the idea that alternatives chosen in a collective choice situation constitute a *final contract* and hence would be closed under re-consideration. As Schwartz (1990) shows (Theorem 4), $TEQ(\cdot)$ is the unique tournament solution $S(\cdot)$ satisfying the following three axioms:

1. **Retentiveness** Nothing in $X \setminus S(T)$ S-contests anything in $S(T)$. Formally, $S(T)$ is $D(S, T)$-retentive for $T \in \mathcal{T}$.
2. **Predictive Strength** If a proper subset $Y \subset S(T)$ is $D(S, T)$-retentive, then $S(T) \setminus Y$ is also $D(S, T)$-retentive, for all $T \in \mathcal{T}(X)$.
3. **Inclusiveness** If $Y \subseteq X$ is $D(S, T)$-retentive in X, then $Y \cap S(T) \neq \emptyset$, for all $T \in \mathcal{T}(X)$.

3 The *l*-contesting banks set

Using the contestation relation, the Banks set may be successively refined by requiring that "winners" bear on "almost winners:" just as the Banks set is the set of points that Banks-contest *some point*, we may define ${}_2B(T)$ as alternatives that Banks-contest *a Banks point*. The Banks set is, in essence, the application of the Banks-contestation relation to the set of alternatives under consideration. We may continue, however, and consider the l^{th}-application of the Banks-contestation relation, as follows: $x \in {}_lB'(T)$ iff $\exists y \in {}_{(l-1)}B'(T)$ such that $xD(B, T)y$, where ${}_0B'(T) = X$. Hence, $B(T) = {}_1B'(T)$. Note that ${}_lB'(T) \subseteq B(T)$ for all $l \geq 1$, for all $T \in \mathcal{T}$.

As defined, ${}_lB'(\cdot)$ is not a tournament solution, because it may be empty. But it may only be empty when there is a clear "best" alternative (i.e. a Condorcet winner). It may be shown that if $Cond(T) = \emptyset$. Then ${}_lB'(T) \neq \emptyset$ for all $l \geq 1$. Hence, for $T \in \mathcal{T}(X)$, define a modified version– the "l-contesting Banks set" – ${}_lB(T)$ as follows:

[4] This is an abuse because according to Definition 2.2, the domain of a tournament solution is always non-empty.

Definition 3.1 $x \in {}_lB(T)$ iff (1) $\exists y \in {}_{l-1}B'(T)$ such that $xD(B,T)y$ or (2) $x = Cond(T)$. Let ${}_\infty B(T) = \bigcap_{k=0}^{\infty} {}_kB(T)$ with the convention that ${}_0B(T) = X$.

Clearly then, $|{}_lB(T)| \geq 1$ for all $l \geq 0$. It may easily be shown that ${}_lB(T) \subseteq {}_{l-1}B(T)$ for all $l \geq 1$. Note the similarity to the $\boldsymbol{l^{th}}$ **Banks set**, $B^l(T)$, which is the l^{th} application of $B(\cdot)$ to T: $B^l(T) = B(T|_{B^{l-1}(T)}), l \geq 1$, where $B^0(T) = X$. The **iterated Banks set**, denoted $B^\infty(T)$, is the intersection of all l^{th} Banks sets: $B^\infty(T) = \bigcap_{k=0}^{\infty} B^k(T)$. The iterated Banks set may be motivated as a tournament solution if, after refining alternatives to the Banks set, groups ignore alternatives not in the Banks set in subsequent refinements. On the contrary, the l-contesting Banks is obtained by considering the *entire* tournament and refines group choice by requiring ever-more stringent "immunity from unseating" of alternatives.

To investigate the relationship between ${}_lB(T)$ and $B^l(T)$, the following will be useful.

Lemma 3.2 *Let* $y \in B^l(T)$. *Then* $B(T|_{T^{-1}(y)}) \subseteq B^l(T)$.

Proof The proof is by induction. That $y \in B(T) \Rightarrow B(T|_{T^{-1}(y)}) \subseteq B(T)$ follows from the definition of $B(T)$. Let $y \in B^{l+1}(T)$. Then $y \in B^l(T)$ and so by inductive hypothesis, $b \in B^l(T)$ for $b \in B(T|_{T^{-1}(y)})$. Then $b \in B(T|_{T^{-1}(y)}) \cap B^l(T)$ implies that there exists $\{y_1, y_2, \ldots, y_k\} \in T^{-1}(y) \cap B^l(T)$ such that the sequence $b, y_1, y_2, \ldots, y_k$ is (i) transitive and (ii) externally stable in $T^{-1}(y) \cap B^l(T)$.

I claim that $b, y_1, y_2, \ldots, y_k, y$ is (i) transitive and (ii) externally stable in $B^l(T)$. The proof of this claim is as follows. We may write $B^l(T) = A \cup y \cup C$ where $A \subseteq T^{-1}(y)$ and $C \subseteq T(y)$. Now (i) by construction $\{b, y_1, y_2, \ldots, y_k\}$ beats y (under the T relation) and; (ii) $b, y_1, y_2, \ldots, y_k$ is externally stable in $T^{-1}(y)$ and y beats all of $T(y)$ (under the T relation) by definition. Hence, $b \in B^{l+1}$, as required. □

Proposition 3.3 $B^l(T) \subseteq {}_lB(T)$ *for all* $T \in \mathcal{T}(X), l \geq 1$.

Proof By induction: $B^1(T) = {}_1B(T) = B(T)$, by definition.

Let $x \in B^{l+1}$. If $Cond(T) \neq \emptyset$, then $Cond(T) = {}_lB(T) = B^l(T)$ and there is nothing to show. If $Cond(T|_{B^l(T)}) = \emptyset$, then there exists $y \in B^l(T)$ such that $x \in B(T|_{T^{-1}(y) \cap B^l(T)})$. Now suppose by way of contradiction that $x \notin {}_{l+1}B(T)$. Then $x \in B(T|_{T^{-1}(y) \cap B^l(T)}) \nsubseteq B(T|_{T^{-1}(a)})$ for all $a \in {}_lB(T)$. By the inductive hypothesis, $y \in {}_lB(T)$, so in particular, $B(T|_{T^{-1}(y) \cap B^l(T)}) \nsubseteq B(T|_{T^{-1}(y)})$. By the (contrapositive of the) Aïzerman property [that $B(\cdot)$ satisfies the Aïzerman property may be seen, for example, in Laslier (1997, 7.1.3(v))], $B(T|_{T^{-1}(y)}) \nsubseteq B^l(T) \cap T^{-1}(y)$. Clearly $B(T|_{T^{-1}(y)}) \subseteq T^{-1}(y)$ and by Lemma 3.2 $B(T|_{T^{-1}(y)}) \subseteq B^l(T)$, so that $B(T|_{T^{-1}(y)}) \subseteq B^l(T) \cap T^{-1}(y)$, contradicting the supposition. □

Proposition 3.3 establishes that the iterated Banks set, $B^l(T)$ is always contained in ${}_lB(T)$. However, the two families of tournament solutions are distinct as the following example shows:

Fig. 1 The tournament used in Example 3.4. A graphical illustration is given on the right, using the convention that the vertical height on the page represents dominance

- $T(a) = \{b\}$
- $T(b) = \{v, w, x, y, z\}$
- $T(v) = \{a, x, z\}$
- $T(w) = \{a, v, y\}$
- $T(x) = \{a, w, y, z\}$
- $T(y) = \{a, v, z\}$
- $T(z) = \{a, w\}$.

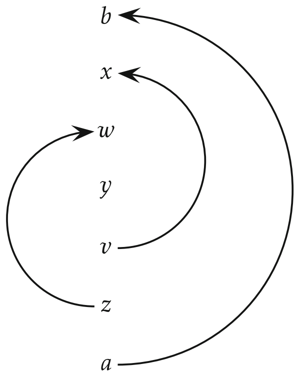
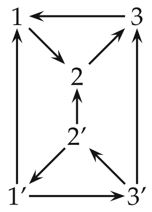


Fig. 2 A tournament T (left) showing that ${}_\infty B(T)$ does not form a $D(B, T)$-cycle and also showing ${}_\infty B$ does not satisfy monotonicity nor predictive strength ($D(B, T)$ relation given on right)



- $1D(B,T)2$
- $2D(B,T)3$
- $3D(B,T)1$
- $1'D(B,T)3'$
- $2'D(B,T)1'$
- $3'D(B,T)2'$
- $1D(B,T)2'$
- $1D(B,T)3'$
- $2D(B,T)3'$
- $2D(B,T)1'$
- $3D(B,T)2'$
- $3D(B,T)1'$

Example 3.4 (${}_2B(T) \setminus B^2(T) \neq \emptyset$) Let $X = \{a, b, v, w, x, y, z\}$ and T given as in Fig. 1. Then $B(T) = \{v, w, x, y, a, b\}$. Further, $y \in B(T^{-1}(a))$ implies $y \in {}_2B(T)$. But $B(T|_{B(T)}) = \{a, b, v, w, x\}$. Hence, $y \in {}_2B(T)$ but $y \notin B^2(T)$.

It may be shown that ${}_lB(T)$ is $D(B, T)$-retentive, for $T \in \mathcal{T}$. Since ${}_lB(T) \subseteq {}_{l-1}B(T)$ (for $l > 0$), it is also the case that ${}_lB(T)$ is $D({}_lB, T)$-retentive. It may also be shown that ${}_lB(T)$ is externally stable with respect to $T|_{{}_{l-1}B(T)}$.

Next I turn attention to the limiting set of contestation, ${}_\infty B$, and its structure. From the definition of ${}_\infty B(T)$ it follows that $\forall x \in {}_\infty B(T)$, $\exists y_x \in {}_\infty B(T)$ such that $xD(B, T)y_x$ if $Cond(T) = \emptyset$. Likewise, if $Cond(T) = \emptyset$, then by the non-emptiness of the Banks set it may be shown that $\forall y \in {}_\infty B(T)$, $\exists x_y \in {}_\infty B(T)$ such that $x_y D(B, T)y$. Hence, in the absence of a Condorcet winner, everything in ${}_\infty B(T)$ both "$D(B, T)$s" and "is $D(B, T)$ed" by something else in ${}_\infty B(T)$, respectively. However, ${}_\infty B(\cdot)$ does not always form a $D(B, \cdot)$-cycle, as can be seen in the tournament given in Fig. 2. In this example, ${}_\infty B(T) = \{1, 2, 3, 1', 2', 3'\}$, which does not form $D(B, T)$-cycle.

Next, I explore the relationship between ${}_\infty B(\cdot)$, and $TEQ(\cdot)$.

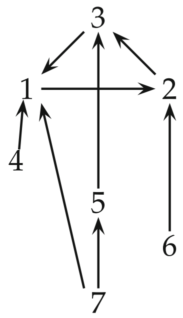
Lemma 3.5 *$TS(D({}_\infty B, T)) \subseteq {}_\infty B(T)$ for all $T \in \mathcal{T}(X)$.*

Proof If $Cond(T) \neq \emptyset$, the result holds trivially.

Let $x \in TS(D({}_\infty B, T))$. Then there exists $\Gamma \subseteq X$ such that (*i*) $\forall a \in \Gamma$, ${}_\infty B(T|_{T^{-1}(a)}) \subseteq \Gamma$ (i.e. Γ is $D({}_\infty B, T)$-retentive) and (*ii*) Γ is minimal (w.r.t. set inclusion) to have this property.

Fig. 3 ${}_{\infty}B$ does not satisfy the Aïzerman property

- $T(1) = \{2, 5, 6\}$
- $T(2) = \{3, 4, 5, 7\}$
- $T(3) = \{1, 4, 6, 7\}$
- $T(4) = \{1, 5, 6, 7\}$
- $T(5) = \{3, 6\}$
- $T(6) = \{2, 7\}$



There are two cases to consider:

1. $|\Gamma| = 1$: then $\Gamma = x$. Hence $T^{-1}(x) = \emptyset$ which implies $Cond(T) = x$, and the result holds trivially.
2. $|\Gamma| > 1$: by assumption, ${}_{\infty}B(T|_{T^{-1}(y)}) \subseteq \Gamma$ for all $y \in \Gamma$. Suppose, by way of contradiction, that there exists some $y \in \Gamma : y \notin {}_{\infty}B(T)$. Then for all $z \in {}_{\infty}B(T)$, it is not the case that $yD(B, T)z$. Because ${}_{\infty}B(T) \subseteq B(T)$, we have $\forall z \in {}_{\infty}B(T)$, $\neg yD({}_{\infty}B, T)z$. This in turn implies that $\Gamma \setminus y$ is $D({}_{\infty}B, T)$-retentive, contradicting the minimality of Γ (item *ii* from above). □

Theorem 3.6 ${}_{\infty}B(\cdot)$*:*

(a) satisfies Retentiveness
(b) satisfies Inclusiveness
(c) does not satisfy the Predictive Strength axiom

Proof Let $|X| > 1$. (a): Because ${}_{l}B(T) \subseteq {}_{l-1}B(T)$ for $T \in \mathcal{T}(X)$ and $l \geq 1$, ${}_{l}B(T)$ is $D({}_{l}B, T)$-retentive.

(b): Let $Y \subseteq X$ be $D({}_{\infty}B, T)$-retentive. Then for all $y \in Y$, ${}_{\infty}B(T|_{T^{-1}(y)}) \subseteq Y$. Since Y is finite, there exists a minimally retentive subset $\Gamma \subseteq Y$. By the definition of top-set, $\Gamma \subseteq TS(D({}_{\infty}B, T))$. By Lemma 3.5, $TS(D({}_{\infty}B, T)) \subseteq {}_{\infty}B(T)$, so we have $\Gamma \cap {}_{\infty}B(T) \neq \emptyset$.

(c): A counter example can be found in Laslier (1997, p. 109). Let T be given as in Fig. 2. Then ${}_{\infty}B(T) = \{1, 2, 3, 1', 2', 3'\}$. However, $\{1, 2, 3\}$ is $D({}_{\infty}B, T)$ retentive, where $\{1', 2', 3'\}$ is not (in particular, $1D({}_{\infty}B, T)1'$). □

Claim 1 ${}_{\infty}B(\cdot)$ *does not satisfy the Aïzerman property.*

The example given in Fig. 3 [taken from (Laslier 1997, p. 110) [5]] provides a counterexample.

${}_{4}B(T) = {}_{\infty}B(T) = \{1, 2, 3\}$ but ${}_{2}B(T|_{X\setminus\{4\}}) = {}_{\infty}B(T|_{X\setminus\{4\}}) = \{1, 2, 3, 5, 6, 7\}$.

Claim 2 ${}_{\infty}B$ *is not monotonic.*

[5] I am grateful to an anonymous referee for suggesting the counterexamples used in Claims 1 and 2.

A counterexample can be found in Fig. 2. As stated before, ${}_{\infty}B(T) = \{1, 2, 3, 1', 2', 3'\}$. But reversing the edge from $1'$ to $3'$ (call the resulting tournament $T^{\{1',3'\}}$), results in ${}_{\infty}B(T^{\{1',3'\}}) = \{1, 2, 3\}$. Hence, by adding support to $1'$, it is no longer in ${}_{\infty}B$.

4 Conclusion

Recently, an elusive open problem concerning the properties of TEQ was answered in the negative. Brandt et al. (2011) shows that TEQ is not pairwise intersecting, and hence does not satisfy SSP and is not monotonic (Laffond et al. 1993). This important result will surely influence future work on tournament solutions. In light of this result, the present work may be put into perspective: ${}_{\infty}B$ does not satisfy monotonocity nor the strong super set property; neither does TEQ.[6] In fact, no tournament solution contained in the Banks set and a super-set of TEQ is pair-wise intersecting as a corollary of Brandt et al. (2011) and hence the family of tournament solutions introduced in Brandt et al. (2010a) does not satisfy monotonicity nor SSP for any S with $TEQ(T) \subseteq S(T) \subseteq B(T) \forall T \in \mathcal{T}$ [specifically, Brandt et al. (2010a, Theorem 1 (ii)) is vacuous for any $S \subseteq B$].[7]

Another operator useful for examining and constructing tournament solutions is the top-set (the union of minimal retentive sets of a binary relation). Indeed, Schwartz (1990), Laslier (1997, Chap. 7) and Brandt et al. (2010a) demonstrate that the contestation relation and the union of minimally retentive sets with respect to the contestation relation prove a powerful "toolbox" for examining choice from tournaments. Brandt et al. (2010a) examine the application of the top-set to the contestation of arbitrary tournament solutions and find, surprisingly, that repeated application of the top-set to the contestation relation results in $TEQ(T)$ for any S (Brandt et al. 2010a, Theorem 2). While the Brandt et al. (2010a) result is powerful, this paper suggests that the strength of their result lies not in contestation per se, but rather in the minimality of retentive sets. This raises the natural question: what are the properties of the top-set of other binary relations obtained from tournament solutions?

In this paper, I explored further uses of the contestation relation for (1) partially separating the effect of the top-set operator in earlier studies of contestation and (2) defining new tournament solutions. In particular, a family of new tournament solutions, ${}_{l}B$, were introduced, the limit of which (${}_{\infty}B$) was investigated and found not to satisfy monotonicity (Fig. 2) nor Aïzerman (Fig. 3). The usefulness of the exercise, however, lies not necessarily in the resulting tournament solution as a method of group choice, but in establishing the limits of taking cooperative re-consideration seriously; the present work shows the limits and limitations of thinking of collective choice from tournaments by iteratively removing from consideration contested alternatives.

[6] A tournament solution $S(\cdot)$ satisfies the strong superset property (SSP) if for all $T \in \mathcal{T}(X), S(T) \subseteq Y \subseteq X \Rightarrow S(T|_Y) = S(T)$. Clearly, SSP implies Aïzerman.

[7] This can be shown by noting that $Ret_{D(S,T)}$ being pair-wise intersecting implies $Ret_{D(B,T)}$ is pairwise intersecting for any $S \subseteq B$ and Brandt et al. (2011) shows there exist T for which the latter does not hold.

Results presented here, when taken with the results of Brandt et al. (2010a, 2011), suggest an examination of the top-set of binary relations other than the contestation relation. This work leaves open the study of other binary relations besides the contestation relation and the top-sets thereof [for example, as in Moser et al. (2009)].

References

Banks JS (1985) Sophisticated voting outcomes and agenda control. Soc Choice Welf 1(4):295–306

Brandt F, Brill M, Fischer F, Harrenstein P (2010a) Minimal retentive sets in tournaments. In: van der Hoek K, Luck S, (eds) Proceedings of 9th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2010), Toronto

Brandt F, Chudnovsky M, Kim I, Norin S, Scott A, Seymour P, Thomassé S, Montpelier F (2011) A counterexample to a conjecture of Schwartz. Soc Choice Welf (forthcoming)

Brandt F, Fischer F, Harrenstein P (2009) The computational complexity of choice sets. Math Log Quart 55(4):444–459

Brandt F, Fischer F, Harrenstein P, Mair M (2010) A computational analysis of the tournament equilibrium set. Soc Choice Welf 34(4):597–609

Dutta B (1990) On the tournament equilibrium set. Soc Choice Welf 7(4):381–383

Houy N (2009) Still more on the tournament equilibrium set. Soc Choice Welf 32(1):93–99

Laffond G, Laslier J, Le Breton M (1993) More on the tournament equilibrium set. Mathematiques et Sci Humaines 31(123):37–44

Laslier J-F (1997) Solutions and majority voting. Springer, Berlin

Moser S, Patty JW, Penn EM (2009) The structure of heresthetical power. J Theor Polit 21(2):139–159

Schwartz T (1990) Cyclic tournaments and cooperative majority voting: a solution. Soc Choice Welf 7(1):19–29