The Structure of Heresthetical Power

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Abstract

This paper considers manipulation of collective choice – in such environments, a potential alternative is powerful only to the degree that its introduction can affect the collective decision. Using the Banks set (Banks [1985]), we present and characterize alternatives that can, and those that can not, affect sophisticated collective decision making. Along with offering two substantive findings about political manipulation and a link between our results and Riker's concept of *heresthetic*, we define a new tournament solution concept that refines the Banks set, which we refer to as the *heresthetically stable set*.

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1 Introduction

William Riker defined *heresthetic* as the art of political strategy. In *The Art of Political Manipulation*, he describes three aspects of heresthetic: (1) *agenda control*, (2) *strategic voting*, and (3) *manipulations of dimensions*.[1] The first two of these aspects – particularly strategic/sophisticated voting are well understood – in particular, the results of Farquharson [1969], McKelvey and Niemi [1978], Shepsle and Weingast [1984], and Banks [1985] (among others) have provided us with an impressive level of understanding about the strategic incentives of both voters and agenda-setters in situations in which the set of alternatives to be voted upon is predetermined.

In this paper we utilize the concept of the Banks set (Banks [1985]) to formally characterize the power of heresthetics. Technically, the Banks set equals the set of policy outcomes achievable through a process of sophisticated voting over an *externally stable agenda* of alternatives. External stability of an agenda implies that the agenda is unamendable, or unmanipulable. Thus, the Banks set alone captures the first two aspects of heresthetic; it represents the collection of all policy outcomes that a strategic agenda-setter can attain when people vote sophisticatedly under an amendment agenda. Our goal in this paper is to examine the effect of the third aspect of heresthetic – "manipulation of issue dimensions" – on this set of feasible outcomes. In particular, by examining the effect on the Banks set of introducing or removing alternatives into a collective choice situation, we are able to capture each of the three aspects of heresthetic. The approach utilized here could be applied to other solution concepts (many of which are discussed below, particularly in Section 6). We restrict our attention to the Banks set because it is the most refined solution concept that is consistent with the first and second components of Riker's definition of heresthetic. Our approach advances the understanding of political manipulation in two important ways.

First, we are able to provide two substantive findings about heresthetical power. While

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[1]This definition of heresthetic is drawn from Riker [1986], pp. 147-8.

these results follow from known theoretical results (as we discuss in more detail later), our approach is constructive and intended to elucidate the important question of how one can manipulate collective choice with sophisticated individuals. Our main substantive findings are:

1. The introduction of a new alternative can eliminate an existing alternative *only if* the newly introduced alternative can itself be the chosen outcome in the new collective choice situation (Theorem 3), and

2. One can introduce a new alternative that enlarges the set of possible collective choices, *even when* the new alternative itself can never be a chosen outcome in the new collective choice situation (Example 1).

In terms of the heresthetic and political manipulation generally, these findings speak to the risks and opportunities facing prospective "herestheticians." As we discuss in the conclusion, our findings illuminate a trade-off faced by any individual who seeks to engage in the manipulation of dimensions. When the individuals in question are strategic, introducing a new alternative in order to preclude the selection of a preexisting one *necessarily* involves some risk: namely, the introduced alternative may actually be selected. On the other hand, if a heresthetician seeks merely to enlarge the set of alternatives that may be collectively chosen, this can sometimes be accomplished by the introduction of an alternative that, in both a colloquial and a formal sense, is innocuous. In the end, completely precise prediction – in the sense of offering forth a unique alternative as the predicted outcome – is usually an unattainable ideal within abstract models of collective choice. As is common with any political situation of interest, the devil is in the details. Though he does not reference set-valued solution concepts in his works on the topic, Riker actually discusses this point.[2] In general, achieving a unique prediction in a specific setting re-

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[2]Specifically, see Riker [1986], Chapters 7 & 12. Throughout *The Art of Political Manipulation*, Riker emphasizes the knowledge of intricate details possessed by successful herestheticians and the ignorance of these details by those who were less successful manipulators.

quires, first and foremost, details about institutions above and beyond the theoretically clean and normatively appealing assumption of majority rule.[3]

At an even deeper level, as our approach in this paper makes clear, allowing for the set of alternatives under consideration to be subject to heresthetical maneuvers highlights the essence of Riker's introduction of the heresthetic as an art, rather than as a science (Riker [1986], p. ix). Specifically and as a practical matter, the third component – manipulation of dimensions – is tricky, to say the least. Part of the difficulty is knowing others' preferences about a potential dimension prior to its introduction. A second difficulty is understanding how the introduction of a new alternative will specifically affect the collective choice process; what outcome will be chosen after the manipulation of dimensions may be unclear precisely because the opportunity for dimension manipulation is present. This highlights once again the trade-off indicated by our two main results: certainty following from heresthetical maneuvers (*e.g.*, precluding some preexisting alternative with certainty) generally comes at some cost (the alternative introduced to preclude may ultimately be chosen itself).

A second advancement offered by this paper is more technical in nature. Our approach is the first (at least of which we are aware) formalization of Riker's widely-cited and discussed notion of political manipulation. This formalization provides the basis for understanding, in a precise sense, the characteristics of alternatives that are *fundamental* to collective choice. Formally, we provide a characterization of the alternatives that in some way are required for sophisticated deliberation to yield any particular outcome; and we already know that each of these particular outcomes will necessarily lie within the Banks set. The Banks set is the most discriminating of the well-known social choice solution concepts. It is contained within the uncovered set (Miller [1980]) and the Pareto set. Accordingly, in spatial settings, it is centrally located and bounded in size (McKelvey [1986], Feld, Grofman and Miller [1989]). It can be quite difficult to compute, however

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[3]A few examples of such institutional details would include individuals' proposal powers, rights to speak, rules governing whether votes are recorded, *etc.*

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(Woeginger [2003], Hudry [2004]).

More to the point, the Banks set is often sensitively dependent upon elements *outside* the Banks set.[4] The principal justification of the Banks set as a solution concept (*i.e.*, a predictive tool) for collective choice situations is that a collective decision should be consistent with a process of collective choice (*i.e.*, an agenda) such that, at the end of the process, the resulting outcome is immune from further strategic behavior by the individuals involved in making the decision. This justification implies that the solution concept may depend on – and the analyst should be interested in – (some) alternatives that are *not* chosen. Furthermore, as we demonstrate in this paper, some of these alternatives are not in the Banks set itself. Given the fact that the definition of the Banks set depends on elements outside of the Banks set, this may not be surprising. However, viewed from the other direction, our results also characterize alternatives that are *irrelevant* for collective choice. While one might disagree about the proper definition of the level of "power" to assign to some alternatives, this subset of the alternative space is clearly power*less*.

The paper proceeds as follows. Some related work is reviewed in Section 2. The theoretical setup is provided in Section 3 and analyzed in Section 4. Section 5 presents five examples intended to illustrate the tightness of the characterizations obtained in Section 4. In Section 6 we discuss the extension of our results to the definition of a *heresthetically stable set*. Finally, Section 7 concludes.

2   Related Theoretical Literature

Miller [1980] defines and characterizes the uncovered set, most importantly showing that uncovered points are precisely those that can beat all other alternatives in at most two steps. He studies the uncovered set using graph theoretic tools and shows a nesting re-

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[4]As discussed in detail in Laslier [1997], the Banks set is not *independent of the losers* and does not satisfy the *strong superset property*. These properties of the Banks set are necessary conditions for the second of our substantive findings (highlighted in Example 1), but neither describe the nature of the Banks set's sensitivity nor constrain the set of alternatives upon which the Banks set may depend, as we do in Theorem 2.

lationship among various solution concepts: the core is a subset of the uncovered set, which is in turn a subset of the top-cycle set. Miller assumes that the set of alternatives is finite and that the majority preference relation is strict: both assumptions are maintained in this paper. However, in recent work, Penn [2006b,a] relaxes both of these assumptions and shows that the equivalence of several different definitions of the uncovered set disappears when preferences are not strict.

Moulin [1986] considers various solution concepts and provides an axiomatization of Miller's uncovered set. He compares solution concepts including (1) the uncovered set, (2) Copeland winners (those alternatives that are majority preferred to the most other elements) and (3) alternatives that are derived from the sophisticated voting algorithm defined in Shepsle and Weingast [1984]. He concludes that the latter is a reasonable tournament solution concept. In this way, the results of this paper represent an extension of Moulin's work.

Huang and Li [1987] consider essentially the converse of our question in the realm of uncovered points: they examine when a subset of uncovered points may be "toppled" (covered) by the addition of alternatives. By embedding a given tournament in a super-tournament, they show how adding alternatives can (in almost all cases) render an arbitrary number of previously uncovered points covered. Their work essentially builds on Bridgland and Reid [1984] who also consider the stability of uncovered points. A main focus of our work is the stability of elements of the Banks set with respect to the other alternatives under consideration. In particular, we examine what happens to the Banks set as alternatives are removed and added.

Most closely related to the substance and approach of this paper are the works of Miller, Grofman and Feld [1990a,b]. Miller et al. were among the first to consider the Banks set in detail.[5] They study the internal structure of the Banks set and show that in the

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[5]Indeed, Miller et al. dubbed the Banks set the "Banks set." Other important and related works during the same time period include Austen-Smith [1987], Banks and Bordes [1988], Dutta [1988], and Schwartz [1990].

absence of a Condorcet winner the Banks set always contains a cycle containing exactly the points in the Banks set. Further, they show that if there is no Condorcet winner, the Banks set is never *minimally* externally stable. That is, there is always some Banks point that could be removed from the Banks set without affecting the external stability of the Banks set. Our work directly complements theirs in the sense that we study the *external* structure of the Banks set.

Dutta, Jackson and Le Breton [2001, 2002, 2004] investigate the endogenous formation of a set of alternatives prior to strategic selection from the set of alternatives in a variety of settings. Their concern is similar to ours: the essence of collective choice is not only what is chosen, but what is chosen *from*. DJL define the *candidate stable set* (henceforth *CS*), which contains all of the alternatives (candidates) who wish to run in an election, given that the other candidates in *CS* (and only those candidates) are remaining in the election.

Finally, Fey [2008] considers the discriminating power of the uncovered and Banks sets in large tournaments. Fey shows that as the size of the tournament (*i.e.*, as the exogenously determined number of alternatives) grows large, the Banks set (and hence uncovered set) almost surely encompasses the entire set of alternatives. Accordingly, Fey suggests that the value of the Banks set as a predictive tool is questionable in situations with large numbers of alternatives. Instead, Fey suggests that the Copeland score of an alternative (the number of alternatives that it defeats) is a better basis for prediction since, unlike membership in the Banks set, the limiting distribution of Copeland scores in a randomly drawn tournament is *not* uniform. Although an interesting and important topic, we do not address the question of asymptotic collective choice in this paper.

3 Theory

We consider a collective choice situation in which an outcome x must be chosen from some finite set of $K \geq 3$ alternatives, X. We denote the binary majority preference relation by $T \subset X^2$ and assume that T is a *tournament*: a complete and antisymmetric binary relation on X. The set of all tournaments on a set of K alternatives is denoted by $\mathcal{T}_K$. The set of alternatives that defeat any alternative x are denoted by $T(x)$ and the set of alternatives that are defeated by x are denoted by $T^{-1}(x)$. For any pair of alternatives $x, y \in X$ and integer $k \geq 1$, we write xT^ky if the length of the shortest path between x and y under T is k. For any tournament T on a set of alternatives X and any integer $k \geq 1$, let $A^k(X, T)$ denote the kth power of T on X: that is, $A^k(X, T)$ is the set $Y \subseteq X$ such that for $y \in Y$, $\forall x \in X$, $\exists j_x$ such that $yT^{j_x}x$ for some $j_x \leq k$ – this is the set of points that can reach *all* other points in at most k steps. For any set X and any subset $Y \subset X$, let $X_{-Y} \equiv X \setminus Y$ and, for any tournament T on X, let $T|_Y$ denote the tournament induced by T on Y.[6] For our purposes, a sequence in X is any (finite) ordered subset of X. Sequences are denoted by the $\sim$ superscript (*e.g.*, $\tilde{x}$, $\tilde{y}$, and so forth).

3.1 Tournaments

Tournaments are frequently (and equivalently) represented as complete, asymmetric directed graphs., with X being equated with the set of vertices, and two vertices $x, y \in X$ being connected only if $(x, y) \in T$. Tournaments have been studied at least since the 1960's, in connection with a wide range of topics from majoritarian voting to ranking of participants in competitions. They are an important tool for studying collective choice situations because, when the number of alternatives is finite and majority preference is strict, constructing a social choice prediction is equivalent to the problem of selecting a set of "winners" from a tournament. In such applications, it is generally assumed that

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[6]In graph theory, $T|_Y$ is referred to as a subtournament. Formally, for any $x, y \in X$, $(x, y) \in T|_Y$ if and only if $(x, y) \in (Y \times Y) \cap T$.

each vertex of the tournament represents a feasible alternative. Under this assumption, so long as the number of voters is odd and each voter's preferences over the alternatives is strict, the resulting majority preference relation may be represented by a tournament, using the convention that $(x, y) \in T$ if and only if x is majority preferred to y.

Much attention has been paid to various solution concepts and their properties in tournament settings because tournaments provide a concise way to visualize various solution concepts (such as the Banks set, the Uncovered set, Copeland winners, etc.). Two excellent book-length introductions to this literature are offered by Moon [1968], who focuses on ranking all alternatives, and Laslier [1997], who proves and provides a reference for many results about solution concepts for majority voting tournaments.

Remark 1 It is important to note that, although the assumption of an odd number of voters with strict transitive preferences insures that the majority preference relation is a tournament, once one fixes the number of voters there exist tournaments that do not represent *any* majority preference relation for a group of n individuals with transitive preferences. In particular, if the number of alternatives, K, exceeds $n - 2$, then there exist tournaments in $\mathcal{T}_K$ that are inconsistent with the majority preference relation generated by *any* profile of n linear orders. This fact suggests another reason that the results of Fey [2008] do not jibe with the usual portrayal of the uncovered and Banks sets as being strict subsets of the set of alternatives.[7]

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[7]Fey discusses this point in his paper (pp. 14-15) and describes the tension as being due to the restrictions imposed on admissible tournaments by the usual assumption of spatial preferences, a point that is clearly important. An interesting question, particularly when juxtaposed with the results of McGarvey [1953], Bell [1981], Bogomolnaia and Laslier [2007] and others, is how much restriction is obtained by assuming a fixed number of voters, relative to that obtained by assuming that individual preferences are consistent with a distance-based utility function in L-dimensional Euclidean space. (Note that the result will depend on L.) In this paper, we are less interested in individual preferences *per se*, though it clearly represents an interesting and relevant topic for future work, especially given the explicit use of the term "dimension" in Riker's description of heresthetic.

3.2 External Stability and Chains

Given a tournament T, a subset $Y \subseteq X$ is

1. *externally stable* if for each $z \in X_{-Y}$, there exists $y_z \in Y$ such that $y_z T z$,

2. a *chain* if $T|_Y$ is transitive, and

3. a *maximal chain* if Y is both a chain and externally stable.

Given T, the set of all externally stable sets is denoted by $M(X, T)$, the set of all chains is denoted by $H(X, T)$, and the set of all maximal chains is denoted by $MC(X, T)$. (So, $MC(X, T) = M(X, T) \cap H(X, T)$.)

3.3 The Banks and Uncovered Sets

Extending the work of Shepsle and Weingast [1984] and others, Banks [1985] demonstrated that sophisticated voting in an amendment agenda and external stability jointly imply that alternatives that can be equilibrium outcomes are exactly identified with maximal chains. Accordingly, for any collective choice situation (X, T) and any $x \in X$, the set of all maximal chains leading to x is denoted by $H_{X,T}(x)$. For any tournament T and pair of alternatives $x, y \in X$, we write $xC_T y$ (or "x covers y with respect to T") if xTy and $T(x) \cap T(y) = T(x)$ (*i.e.*, x beats y and any alternative z that beats x beats y as well). We are now in a position to define two important social choice-theoretic notions. Given X and T, the *uncovered set* (Miller [1980]), $UC(X, T)$, is defined as follows:

$$UC(X, T) = \{x \in X : yTx \Rightarrow \exists z \text{ s.t. } xTzTy\} = \{x \in X : \{y \in X : yC_T x\} = \emptyset\},$$

and the *Banks set* (Banks [1985]), $B(X, T)$, is defined as follows:

$$B(X, T) = \{x \in X : H_{X,T}(x) \neq \emptyset\}.$$

The following important fact is well-known.[8]

Theorem 1 (Miller [1980], Shepsle and Weingast [1984], Banks [1985])

$$B(X,T) \subseteq UC(X,T) = A^2(X,T).$$

4   The Role of Policy Winners and Losers

Banks [1985] formalized the notion of the set of sophisticated voting outcomes. The Banks set is the set of alternatives achievable as sophisticated voting outcomes of an amendment agenda. Thus, if the role of a strategic agenda-setter is to determine the order by which all alternatives are successively voted upon, and if voters are assumed to also vote strategically, then the power of an agenda-setter to set policy is only so large as the Banks set.[9]

The Banks set elegantly captures the first two components of the heresthetic (*agenda control* and *strategic voting*). However, membership in the Banks set depends upon the relationship between the item in question and (essentially) every other element under consideration. Understandably, the definition of the Banks set takes the set of alternatives under consideration as a given. Thus, while the Banks set represents an integral part of the heresthetic, it does not capture all of what Riker intended the term to describe. In particular, the third component of the heresthetic, the *manipulation of dimensions*, is quintessentially tied to the set of alternatives under consideration. The goal of this section is to provide a framework for utilizing the concept of the Banks set in order to study the strategic introduction and exclusion of issues in order to manipulate the collectively chosen outcome. We begin by defining two simple types of alternatives that can affect

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[8]As Banks [1985] shows, if $K > 8$, there are tournaments T for which the inclusion stated in Theorem 1 is strict.

[9]Agenda-setting is a broad notion in political economy, encompassing many different institutional settings. Perhaps a more accurate title of the position that Banks examines is "Order-of-voting-setter." However, Penn [2006*b*] extends the definition of the Banks set to the case of an infinite policy space. In this setting an agenda-setter both chooses a subset of alternatives for consideration and orders the subset.

whether or not other alternatives are sophisticated winners: those alternatives that are *necessary* for a current sophisticated winner, and those that, when removed from the set of alternatives, *create* a brand new sophisticated winner. These alternatives will be termed *Banks-necessary* and *Banks-creating* alternatives, respectively. While we conduct our analysis in terms of removing alternatives from consideration, note that this is a difference in appearance only – those points that are Banks-necessary prior to being removed would clearly enlarge the Banks set if added back into the set of alternatives (and vice-versa).

Definition 1 *Given a collective choice situation* (X, T),

- *the set of* Banks-necessary *alternatives is*

$$BN(X, T) = \{x \in X : B(X, T) \setminus B(X_{-x}, T) \neq \emptyset\},$$

- *the set of* Banks-creating *alternatives is*

$$BC(X, T) = \{x \in X : B(X_{-x}, T) \setminus B(X, T) \neq \emptyset\}.$$

In words, an alternative x is Banks-necessary if removing x from the set of alternatives also removes an element y from Banks set. Thus, it is necessary for x to be in the set of alternatives for y to be a sophisticated voting outcome. Clearly, all points in the Banks set are Banks-necessary ($B(X, T) \subseteq BN(X, T)$). Thus, having x in the set of alternatives prevents y from being a sophisticated winner. An alternative x is Banks-creating if removing x from the set of alternatives adds a new element z to the Banks set. The following two theorems provide preliminary characterizations of these sets. Prior to stating these results, we provide the following lemma.

Lemma 1 *For any* $x \notin A^3(X, T)$ *and* $Y \subseteq X$, $Y \in M(X, T) \Leftrightarrow (Y_{-x}) \in M(X, T)$.

Proof: If $x \notin Y$, the result holds trivially. The implication $Y \in M(X, T) \Leftarrow (Y_{-x}) \in$

$M(X,T)$ is obvious. Accordingly, we need to show only that $Y \in M(X,T) \Rightarrow (Y_{-x}) \in M(X,T)$ i.e. (Y_{-x}) is externally stable if Y is externally stable.

For any subset $Y \subseteq X$, let $\bar{T}(Y) \equiv \cap_{x \in Y} T(x)$ denote the set of alternatives that defeat every element of Y under T. Note that $x \in \bar{T}(Y)$ implies $x \notin Y$. Suppose, in order to establish a contradiction, otherwise: there exists $x \notin A^3(X,T)$ and $Y \subseteq X$ such that $Y \in M(X,T)$ ($\bar{T}(Y) = \emptyset$) and $(Y_{-x}) \notin M(X,T)$ (*i.e.*, $\bar{T}(Y_{-x}) \neq \emptyset$). Because $x \notin A^3(X,T)$, there exists an alternative $y_x \in X$ such that y_x covers x *and* covers any alternative $w \in T^{-1}(x)$ (*i.e.*, y_x covers any alternative that x defeats). If Y is externally stable and Y_{-x} is not, then there must exist some alternative $w \in T^{-1}(x) \cap \bar{T}(Y_{-x})$. However, the existence of such an alternative w implies that Y is not maximal, since $y_x C_T w$, so that wTz implies that $y_x Tz$, which implies that $y_x \in \bar{T}(Y)$. This contradicts the supposition that $Y \in M(X,T)$. ∎

The next result establishes that only elements of $A^3(X,T)$ need be considered when constructing externally stable sets. Since every maximal chain is externally stable, the result further implies that $A^3(X,T)$ contains all of the alternatives that affect $B(X,T)$. As mentioned earlier in the paper, this result represents a strict strengthening of our second substantive point – that elements of the Banks set may rely upon alternatives that are not in the Banks set –in the sense that it pins down the set of alternatives that may affect the Banks set.[10]

Theorem 2 $BN(X,T) \subseteq A^3(X,T)$.

Proof: $x \in BN(X,T)$ implies that there exists $y \in B(X,T)$ such that $x \in \tilde{y}$ for any $\tilde{y} \in H_{X,T}(y)$, implying that $\tilde{y} \setminus \{x\} \notin M(X,T)$. Since $\tilde{y} \in M(X,T)$, Lemma 1 (p. 12) then implies that $x \in A^3(X,T)$. ∎

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[10] Recently, this result has been extended by Patty and Penn [2007], who examine the *needing relation*, which essentially characterizes one alternative as relying uniquely upon a second alternative for the first alternative's shortest path in the tournament. They show that this relation has an intimate connection with all solution concepts that refine the uncovered set. Because their analysis is in some sense nested within that contained here, their results also indicate the connection between this paper's analysis and the structure of agenda control in legislatures.

The next result – the basis for the first of the paper's two main substantive findings – states that the set of Banks creating points is contained within the Banks set. Our proof of the result is a simple and direct one that directly references the definition of the Banks set in terms of maximal chains. The result can be demonstrated in a shorter (but we believe less transparent) fashion by appealing to the fact that the Banks set satisfies the Aïzerman property.[11] Put slightly differently, the construction of $BC(X,T)$ and the statement of Theorem 3 jointly imply that the Banks set satisfies the Aïzerman property.

Theorem 3 $BC(X,T) \subseteq B(X,T)$.

Proof: $x \in BC(X,T)$ implies that $\exists H \in MC(X_{-x},T)$ such that $H \notin MC(X,T)$. This implies that $\exists z \in X$ such that zTy for all $y \in H$, and consequently, that $z \in B(X,T)$. However, $z \in X$ and $z \notin X_{-x}$ imply that $z = x$. Thus $x \in B(X,T)$. ∎

Theorem 3 says that for the removal of an alternative to create a new sophisticated winner, the removed alternative must itself be a sophisticated winner. However, Theorem 2 says that for the removal of an alternative to remove a sophisticated winner, that alternative must only be able to defeat any other alternative in three or fewer steps. Thus, it is possible not only for a Banks necessary alternative to not be in the Banks set itself, but also for such an alternative to not even be uncovered. Example 1 (p. 15) in the following section demonstrates this.

5 Some Examples

In this section, we provide three examples that demonstrate in some sense the "tightness" of the results obtained in the previous two sections. In particular, we demonstrate that the relation of both the uncovered set and the Banks set to $BN(X,T)$ and $BC(X,T)$ can not

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[11]For a formal statement and proof of this result and a discussion of the Aïzerman property, see Laslier [1997], Chapter 7.

be summarily described in any more detail.[12] As discussed later in the paper, Example 1 illustrates a situation in which an individual interested in expanding the Banks set is able to do so through the addition of an alternative that itself can not be chosen from the expanded set of alternatives. It also illustrates the potential impact of removing even a covered alternative from the set of choices.

Example 1 $BN(X,T) \setminus UC(X,T) \neq \emptyset$.

Consider the following collective choice situation, (X,T):

- $X = \{v,w,x,y,z\}$,

- zTw,

- yTz, yTv,

- xTz, xTy, xTw,

- wTy, wTv,

- vTz, vTx.

In this example, x covers z. At the same time, the presence of z is necessary for y to be in the Banks set. To see this, note that if z is removed from the set of alternatives, then y is covered by w and thus cannot be in the resulting Banks set. However y is the sophisticated winner of the maximal chain $H = (z,v,y)$. $\triangle$

This next example shows that it is also not the case that $UC(X,T) \subset BN(X,T)$. This implies that the relationship in Theorem 2 may be strict, as $UC(X,T) \subset A^3(X,T)$.

Example 2 $UC(X,T) \setminus BN(X,T) \neq \emptyset$.

Consider the following collective choice situation, (X,T):

- $X = \{t, u, v, w, x, y, z\}$,

- zTy, zTx, zTw,

- yTx, yTv,

- xTw, xTu,

- wTy, wTt,

- vTz, vTx, vTw, vTt,

- uTz, uTy, uTw, uTv,

- tTz, tTy, tTx, tTu.

Note that in this example z is uncovered but not in $B(X,T)$. $B(X,T) = \{y, x, w, v, u, t\}$. Policy y is the sophisticated voting outcome of the maximal chain $H = (x, v, y)$. Policy v is the sophisticated voting outcome of maximal chain $H = (x, t, v)$. By a similar construction, alternatives x and w, and u and t are sophisticated outcomes of maximal chains that do not contain z. Thus, removing z from the set of alternatives does not affect the Banks set. $\triangle$

The following example shows that there are examples in which the inclusion stated in Theorem 3 is strict. Such examples are interesting because they highlight the possibility that elements of the Banks set can have very different degrees of power: some, when removed from consideration, lead to previously unattainable proposals becoming attainable, while others do not. The example constructs an alternative that, when removed, changes the Banks set only in the most obvious of ways (namely, the removed alternative itself is not in the new Banks set).

Example 3 $B(X,T) \setminus BC(X,T) \neq \emptyset$.
Consider the following collective choice situation, (X, T):

- $X = \{w, x, y, z\}$,

- zTy,

- yTx, yTw,

- xTz, xTw,

- wTz.

In this example $B(X, T) = \{z, y, x\}$. However $BC(X, T) = x$; removing x enables w to be a sophisticated voting outcome of the maximal chain $H = (z, w)$. However, because x covers w, removing either z or y does not enable w to be a sophisticated winner. $\triangle$

6 The Heresthetical Set

The results presented above, though not all new in a formal sense, highlight the desirability of discriminating among elements in the Banks set based on their sensitivity to elements outside the Banks set. In this section, we define a new solution concept – the heresthetically stable set – that selects those alternatives least susceptible to the third component of Riker's definition: manipulation of dimensions. In line with the earlier discussion and results, susceptibility to this form of manipulation is judged as a function of an alternative's continued membership in the Banks set in the face of the elimination of other alternatives from the collective choice situation. From a strategic political standpoint, however, a heresthetician is essentially interested in those situations in which his or her most-preferred alternative within the Banks set is *not* in the heresthetically stable set – these are the situations in which the successful "manipulation of dimensions" (*i.e.*, alteration of the choice set) is particularly relevant for political success.

The approach taken in this section depends, essentially, on representing the membership and dependence upon the set $BN(X, T)$ as a binary relation. In words, the approach

is to relate any pair of alternatives x and y in such a way as to capture whether x's membership in the Banks set depends upon the presence of y in the collective choice situation. Perhaps obviously, if this is the case, then $y \in BN(X, T)$ *and* a collective choice of x (at least in accordance with the logic of the Banks set) will occur only if y is not eliminated from the collective choice situation. After defining this relation a solution concept is presented, essentially refining the Banks set to those elements that are "least sensitive" (and some may be completely insensitive) to the deletion of any single alternative.

Formally, for any collective choice situation (X, T), define the binary relation $\beta \subset X^2$ recursively as follows:

$$y \in B(X, T) \ \& \ y \notin B(X_{-x}, T) \Rightarrow x\beta^0 y$$

and

$$x\beta^0 y \ \Rightarrow \ x\beta y,$$
$$y \notin B(X, T) \ \& \ x \in B(X, T) \ \& \ \neg \left[y\beta^0 x\right] \ \Rightarrow \ x\beta y.$$

For any collective choice situation (X, T), the *heresthetically stable set*, denoted by $\mathcal{HS}(X, T)$, is defined as the top set of β (the union of all minimally retentive subsets of X with respect to β).

In words, each alternative in the heresthetically stable set either (1) remains in the Banks set even if any other alternative is removed from the collective choice situation or (2) there is at least one other element of the Banks set that is not in the Banks set if the alternative is removed. Substantively, alternatives in the heresthetically stable set are more robust than other elements of the Banks set: all other elements of the Banks set are strongly dependent on one or more alternatives in a descriptive way. The usual justification of the Banks set is as the set of outcomes that can be chosen by sophisticated voting

over an "amendment-proof" agenda,[13] implying that elements in the heresthetically stable set are effectively supported by a larger set of agendas than elements in the Banks set that are not in the heresthetical set.

The heresthetically stable set is a tournament solution concept (Laslier [1997]) – for any collective choice situation (X, T), $\mathcal{HS}(X, T) \neq \emptyset$, $\mathcal{HS}$ is invariant under tournament isomorphisms, and $\mathcal{HS}(X, T)$ is equal to the Condorcet winner of (X, T) when one exists. As the next theorem shows, the heresthetically stable set refines the Banks set.

Theorem 4 $\mathcal{HS}(X, T) \subseteq B(X, T)$.

Proof: See appendix. ∎

Accordingly, an open question of significant interest is its relation to other tournament solution concepts. While we leave a full investigation of the properties of the heresthetically stable set for future work, some simple examples (drawn from above) illuminate some interesting features of it. Before presenting these, though, it is useful to consider why the admittedly abstract notion of a "top set" serves as the basis of $\mathcal{HS}$. The best example for this is the smallest nontrivial collective choice situation, the Condorcet cycle: $X = \{x, y, z\}$ with $xTyTzTx$. It is well known that $B(X, T) = X$ in this case. Furthermore, it is simple to verify that $iTj \Rightarrow j\beta i$ for all $i, j \in X$. Accordingly, if one considered the alternate definition

$$\mathcal{HS}'(X, T) = \{x \in X : y\beta x \Rightarrow y = x\},$$

it follows immediately that $\mathcal{HS}' = \emptyset$ for the canonical example of cyclic collective preference.[14] The next set of examples provide a cursory illustration of the properties of the heresthetically stable set in the tournaments considered earlier in the paper.

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[13] An agenda $\tilde{x}$ is "amendment-proof" if the sophisticated voting outcome of any agenda beginning with $\tilde{x}$ is equal to that of $\tilde{x}$.

[14] Given some of the discussion that ensues below, it is worth noting at this point that the Condorcet cycle example similarly motivates the use of the top set operator in the definition of the weak uncovered set offered in Laslier [1997].

6.1 The Heresthetically Stable Set in Action: Three Examples

Example 1. Consider again the collective choice situation presented in Example 1. In this collective choice situation $B(X, T) = \{v, w, x, y\}$. The sets of maximal chains for each alternative are

- $H_{X,T}(v) = \{(z, x, v), (x, v)\}$

- $H_{X,T}(w) = \{(v, y, w), (v, w)\}$

- $H_{X,T}(x) = \{(w, x), (y, x), (y, w, x), (z, y, x)\}$

- $H_{X,T}(y) = \{(z, v, y)\}$

- $H_{X,T}(z) = \emptyset$

This implies that the β^0 relation for this (X, T) is as follows:

- $v\beta^0 v,\ v\beta^0 w,\ v\beta^0 y$

- $w\beta^0 w$

- $x\beta^0 v,\ x\beta^0 x$

- $y\beta^0 y$

- $z\beta^0 y$

The β relation is then as follows:

- $v\beta v,\ v\beta w,\ v\beta y,\ v\beta z$

- $w\beta w,\ w\beta z$

- $x\beta v,\ x\beta x,\ x\beta z$

- $y\beta y$

- $z\beta y$

First, consider singletons: $\{v\}$ is not β-retentive ($x\beta v$), $\{w\}$ is not β-retentive ($v\beta w$), $\{y\}$ is not β-retentive ($v\beta y$), $\{z\}$ is not β-retentive ($v\beta z$), but $\{x\}$ *is* β-retentive. The remaining step is to check whether any $D \subset X_{-x}$ can be β-retentive. Such D can not contain v or z, so consider $D = \{w, y\}$. This is not β-retentive, since $v\beta w$ (and $z\beta y$). Accordingly, any β-retentive D with $D \cap X_{-x} \neq \emptyset$ must contain x, implying that (since $\{x\}$ is β-retentive) $\{x\}$ is the unique minimally β-retentive set, so that $\mathcal{HS}(X, T) = \{x\}$. In other words, the notion of heresthetical stability can provide a unique prediction even when there is no Condorcet winner.

This example is important in the following sense: it demonstrates constructively that $\mathcal{HS}(X, T)$ may be a proper subset of the *tournament equilibrium set* (Schwartz [1990]), which is equal to $\{w, x, v\}$ in this example. $\mathcal{HS}(X, T)$ is also smaller than the *minimal covering set* (Dutta [1988]), which is also equal to $\{x, w, v\}$ in this example. In this example, $\mathcal{HS}(X, T)$ is equal to the *weak uncovered set*, defined as the top set of the *1-weak covering relation* (Laffond and Laine [1994]; Laslier [1997]).

Example 2. Considering again the collective choice situation presented in Example 2, recall that $B(X, T) = \{t, u, v, w, x, y\}$. For reasons of space and exposition, we omit the derivation of β in this example. It is relatively straightforward to demonstrate, nonetheless, that the heresthetically stable set, tournament equilibrium set, minimal covering set, and weak uncovered set are each equal to $\{t, u, v\}$. This is particularly interesting not only because it demonstrates that $\mathcal{HS}$ does not strictly refine any of those solution concepts, but also because $B^\infty(X, T) = UC^\infty(X, T) = B^2(X, T) = UC(B(X, T)) = B(X, T)$. Thus, $\mathcal{HS}(X, T)$ accomplishes a stronger refinement than infinite iteration of either the uncovered set or the Banks set.

Example 3. Now consider the collective choice situation presented in Example 3. In this collective choice situation, $B(X, T) = \{x, y, z\}$. The maximal chains for each element are

- $H_{X,T}(w) = \emptyset$

- $H_{X,T}(x) = \{(z,x),(w,x),(z,w,x)\}$

- $H_{X,T}(y) = \{(x,y),(w,y),(w,x,y)\}$

- $H_{X,T}(z) = \{(y,z)\}$

Omitting the calculation of β^0,[15] the β relation for this (X,T) is as follows:

- $x\beta w$, $x\beta x$,

- $y\beta w$, $y\beta z$, $y\beta y$,

- $z\beta w$, $z\beta z$.

Note that $\{x\}$ is β-retentive, and $\{y\}$ is β-retentive, but $\{z\}$ is not β-retentive, since $y\beta z$. Accordingly, $\mathcal{HS}(X,T) = \{x,y\}$. This tournament is interesting for two reasons: first, $\mathcal{HS}(X,T)$ contains exactly two elements, one of which (y) is majority preferred to the other (x). Second, the weak uncovered set in this example is $\{y\}$, while the uncovered, Banks, tournament equilibrium, and minimal covering sets are all equal to $\{x,y,z\}$. Thus, it is not the case that the heresthetically stable set is equal to the weak uncovered set, which is not obvious from looking at their definitions (indeed, one might at first suspect that the two are equal), since they both rely on the notion of a top set and are contained within the uncovered set.

6.2 Open Questions

We believe that the heresthetically stable set represents a first step toward understanding two important aspects of group choice. First, like several other solution concepts (*e.g.*, the tournament equilibrium set and the *bipartisan set* (Laffond, Laslier and Le Breton [1993])), the heresthetically stable set refines the Banks set (and, hence, the uncovered set), and

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[15]It follows from inspection that $\neg\left[w\beta^0 s\right]$ for all $s \in X$.

represents a potentially useful predictive tool for discriminating among alternatives that heretofore have appeared equally "plausible" as collective decision making outcomes. Second, the heresthetically stable set is derived from concerns about the external stability of the Banks set. As noted earlier in the paper, the Banks set is sensitive "to the losers" and, accordingly, is sensitive to the details of the decision-making process (*e.g.*, one or more alternatives that will not be chosen are removed from consideration prior to the operation of the process, as happens frequently in legislatures). The heresthetically stable set has the potential of offering a (set of) solution(s) that are, essentially, "less vulnerable" to changes in such details. In words, the heresthetically stable set represents the set of outcomes that the analyst thinks least likely to be ruled out by alterations in the set of alternatives under consideration.

Formally, there are many questions left to be answered. For the sake of furthering future research, we list three of these here.

1. Is the heresthetically stable set a subset of the tournament equilibrium set? Of the minimal covering set? Of the intersection of the two?

2. How does $\mathcal{HS}(X, T)$ correspond to the idempotent extension of the uncovered set, or *iterated uncovered set*, $UC^\infty(X, T)$?

3. What is the relation between weak covering and heresthetical stability? For example, is $\mathcal{HS}$ always a subset of the weak uncovered set?

7 Conclusion

The world is messy and formal theory attempts to make order out of it – accordingly, sometimes the best that a formal theory can offer are predictions about what will not happen. Perhaps unsurprisingly, our results are concordant with this statement. Our results indicate that, when we shine our investigative light upon the set of alternatives

introduced into an abstract collective choice problem that are consistent with the solution set from the underlying choice situation, much purchase may be gained on which alternatives are required by the solution set and which ones are redundant. Our discussion of these results has been confined to a consideration of the Banks set. We think this focus is appropriate for several reasons. First, the Banks set is consistent with sophisticated behavior in a wide variety of real-world collective choice situations (Miller, Grofman and Feld [1990*a*]). Second, the Banks set is composition consistent (i.e. insensitive to the cloning of one or several alternatives) whereas the Copeland set (*i.e.,* the set of all Copeland winners) and the top-cycle set are not (Laffond, Lainé and Laslier [1996], Laslier [1997]). But most importantly, any set other than the Banks set either excludes sophisticated outcomes or includes non-sophisticated outcomes. Given that Riker explicitly lists sophisticated voting as a central component of heresthetics, any other concept would either weaken or strengthen Riker's original notion.

The external structure of the Banks set is more than just a technical curiosity. Given the presumption of sophisticated behavior in a deliberative body, this structure represents the essential underpinnings of the deliberative processes through which collective choices will be made. Indeed, knowing this structure is necessary to actually confirm when and whether the individual behavior in question is sophisticated. One of the main themes of modern political science is that understanding what is chosen requires understanding *how* it is chosen. More to the point, understanding what characterizes an alternative's ability to affect collective decision-making is perhaps the most important question in politics. Riker's conception of the heresthetician is one who practices the art of manipulation through the introduction (or preclusion) of such alternatives. We think the results of this paper represent a first step toward identifying and understanding the heresthetician's "tools": the alternatives that are essential to any strategic collective choice situation. One of Riker's key insights (Riker [1980]) was that these tools are central to the pursuit of predictability in the face of the unsettling results of Arrow [1951], McKelvey [1976], and

Schofield [1978]. The notion of the heresthetically stable set is intended to capture the set of outcomes that are robust to perhaps the most intriguing component of the heresthetic: manipulation of dimensions.

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Appendix: Proof of Theorem 4

To prove Theorem 4, the following will be useful.

Definition 2 (Iterated Banks Set) *The lth Banks set, $B^l(X,T)$, is the lth application of $B(\cdot,\cdot)$ to (X,T): $B^l(X,T) = B(B^{l-1}(X,T), T|_{B^{l-1}(X,T)})$, $l > 1$, where $B^1(X,T) = B(X,T)$. The* **iterated Banks set**, *denoted $B^\infty(X,T)$, is the intersection of all l^{th} Banks sets: $B^\infty(X,T) = \cap_{k=1}^\infty B^k(X,T)$.*

When T is understood, we write $B^\infty(X)$ for $B^\infty(X,T)$.

Lemma 2 $H \in MC(B^l(X,T), T)$ *implies* $H \in MC(X,T)$, *for $l > 0$.*

Proof: Let $H \in MC(B(X,T), T|_{B(X,T)})$. Denote the elements of H by $\{h_1, h_2, \ldots, h_m\}$ such that $h_i T h_{i+1}$ for $0 < i < m$. Suppose, by way of contradiction, that $\exists x_1 \in X \setminus B(X,T)$ such that $x_1 T h_i$ for all $h_i \in H$. Create a new chain H_1 by adjoining x_1 to H so that $H_1 = \{x_1, h_1, h_2, \ldots, h_m\}$. If H_1 is not maximal in X, then there exists $x_2 \in X \setminus \{B(X,T) \cup x_1\}$ such that $x_2 T h$ for all $h \in H_1$. Since $|X| < \infty$, there exists some finite integer j such that H_j is maximal in X, and by construction, H_j is a chain. Therefore, the top-element of H_j, x_j is in the Banks set. But this contradicts $x_j \notin B(X,T)$. Hence, $H \in MC(B(X,T), T|_{B(X,T)})$ implies $H \in MC(X,T)$.

The proof is completed by noting that $B^l(X,T) = B(B^{l-1}(X,T), T)$ so that $H' \in MC(B^l(X,T), T)$ implies $H' \in MC(B^{l-1}(X,T), T)$. By induction, $H' \in MC(X,T)$. ■

Lemma 2 says that every maximal chain in $B^l(X,T)$ (for $l > 0$) is a maximal chain in X as well. We are now in a position to prove Theorem 4.

Theorem 4 $\mathcal{HS}(X,T) \subseteq B(X,T)$.

Proof: Let $M \subseteq X$ be in the top-set of β. There are two cases to consider:

$|M| = 1$: Suppose, by way of contradiction, that $x \notin B(X,T)$ and $x \in TS(\beta)$. Then there does not exist any Banks point $b \in B(X,T)$ such that $b\beta x$, which implies that for all $b \in B(X,T)$, $x\beta^0 b$. Hence, alternative x is Banks-needed by *every* Banks point. But this is a contradiction: if x is not a Banks point, x cannot be Banks-creating and so $B(X \setminus x, T|_{X \setminus x}) = \emptyset$. But $B(X,T)$ is non-empty for all (X,T) (see, for example, Laslier [1997, 7.1]).

$|M| > 1$: Note that by the definition of the iterated Banks set, $b \in B^\infty(X,T)$ implies $H_{B^\infty(X), T|_{B^\infty(X)}}(b) \subseteq B^\infty(X,T)$. And by Lemma 2, all of $H_{B^\infty(X), T|_{B^\infty(X)}}$ is maximal in X as well. Hence, if $b \in B^\infty(X,T)$ and $x\beta^0 b$, then $x \in B^\infty(X,T)$ (that is, every element of $B^\infty(X,T)$ can only Banks-need other elements of $B^\infty(X,T)$).

Thus, for any $x \notin B(X,T)$ and any $b \in B^\infty(X,T)$ we have $b\beta x$. Hence, by the above arguments we have: if $b \in B^\infty(X,T)$ and $x\beta b$, then $x\beta y$, for any $y \notin B(X,T)$.

Finally, any β-cycle M involving a non-Banks point ($y \notin B(X,T)$, $y \in M$) cannot be in the top-set of β, because $\nexists m \in M$, such that $m\beta b$ for any $b \in B^\infty(X,T)$ (and $B^\infty(X,T) \neq \emptyset$). Therefore, $\mathcal{HS}(X,T) \subseteq B(X,T)$. ∎